Differentiation:

Type 1: Easy Powers

$$x^n \Rightarrow nx^{n-1}$$

bring the power down to the front and subtract one from the power

e.g. $y = 2x^4 \Rightarrow \frac{dy}{dx} = 2(4)x^3 = 8x^3$

Type 2: Harder Powers

$$(f(x))^n \Rightarrow n(f(x))^{n-1} f'(x)$$

bring the power down to the front and subtract one from the power (keep what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket

e.g. $y = (3x^2 - 4)^6 \Rightarrow \frac{dy}{dx} = 6(3x^2 - 4)^5(6x) = 36x(3x^2 - 4)^5$

Type 3: Exponentials

$$e^{f(x)} \Rightarrow e^{f(x)} f'(x)$$

" copy the exponential and then multiply by the derivative of the power

e.g. $y = e^{4x} \Rightarrow \frac{dy}{dx} = e^{4x}(4) = 4e^{4x}$

$$a^{f(x)} \Rightarrow (a^{f(x)}) f'(x) \ln a$$

copy the exponential, multiply by the derivative of the power and also by ln of the base

e.g. $y = 3^{4x} \Rightarrow \frac{dy}{dx} = 3^{4x}(4)\ln 3 = 4 \ln 3 (3^{4x})$

Type 4: Logarithms

$$\ln f(x) \Rightarrow \frac{f'(x)}{f(x)}$$

This goes to a fraction : $\frac{derivative\ of\ argument}{copy\ of\ argument}$

Notice how the ln disappears

e.g. $y = ln\ (2x - 5) \Rightarrow \frac{dy}{dx} = \frac{2}{2x-5}$

Type 5: Trigonometry

$$\sin f(x) \Rightarrow \cos f(x)\, f'(x)$$
$$\cos f(x) \Rightarrow -\sin f(x)\, f'(x)$$
$$\tan f(x) \Rightarrow \sec^2 f(x)\, f'(x)$$
$$\sec f(x) \Rightarrow \sec f(x) \tan f(x)\, f'(x)$$
$$\csc f(x) \Rightarrow -\csc f(x) \cot f(x)\, f'(x)$$
$$\cot f(x) \Rightarrow -\csc^2 f(x)\, f'(x)$$

For each type we change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle

e.g. $y = \sin 4x^2 \Rightarrow \frac{dy}{dx} = \cos 4x^2 \times 8x = 8x\cos 4x^2$

WATCH OUT for trig to powers: write $\sin^m x$ as $(\sin x)^m$ and then proceed as for harder power type

e.g. $y = \sin^4 2x$

Write this as $y = (\sin 2x)^4$

$$\frac{dy}{dx} = 4(\sin 2x)^3(2 \cos 2x) = 8 \cos 2x \sin^3 2x$$

Type 6: Inverse Trigonometry

$$\sin^{-1} f(x) = \arcsin f(x) \Rightarrow \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

This goes to a fraction: $\frac{derivative\ of\ angle}{\sqrt{1-(angle)^2}}$

$$\cos^{-1} f(x) = \arccos f(x) \Rightarrow \frac{-f'(x)}{\sqrt{1-(f(x))^2}}$$

This goes to a fraction: $\frac{-derivative\ of\ angle}{\sqrt{1-(angle)^2}}$

$$\tan^{-1} f(x) = \arctan f(x) \Rightarrow \frac{f'(x)}{1+(f(x))^2}$$

This goes to a fraction: $\frac{derivative\ of\ angle}{1+(angle)^2}$

Notice how the inverse trig disappears for each type

e.g. $y = \sin^{-1} 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6x}{\sqrt{1-(3x^2)^2}} = \frac{6x}{\sqrt{1-9x^4}}$

Product and Quotient Rule:

We use this when we have one of the 6 types multiplied together (product rule) or divided by each other (quotient rule)

Consider the 2 functions u and v:

Product Rule:

$$y = uv \Rightarrow \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

The formula basically says : (**differentiate** 1st function)(copy 2nd function)**+**(copy 1st function)(**differentiate** 2nd function)

Quotient Rule:

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The formula basically says : $\frac{(copy\ denominator)(differentiate\ numerator) - (copy\ numerator)(differentiate\ denominator)}{(denominator)^2}$

Implicit and Parametric:

Implicit:

We use this when we don't have y only on one side and x only on the other side. Instead, we have a function of x and y on one side equal to something else.

Step 1: Differentiate with respect to x. "every time we differentiate a y write $\frac{dy}{dx}$ ". Use rules 1-6 to differentiate.

Step 2: Collect all the $\frac{dy}{dx}$'s on one side

Step 3: Factorise $\frac{dy}{dx}$ out and then solve for $\frac{dy}{dx}$

Parametric:

Given y and x equations in terms of another variable (lets say t). Differentiate both equations separately.

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$\frac{d^2y}{dx^2} = \left(\text{derivative of } \frac{dy}{dx}\right) \times \frac{dt}{dx}$$

Basic Examples

Harder Powers:

$$y = (3x - 2)^5 \text{ . Find } \frac{dy}{dx}$$

$$y = (3x - 2)^5$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the harder powers type. The rule for harder powers is:

"**bring the power down to the front** (and multiply it by the number at the front if there is one), then **subtract one from the power** (keeping what is inside the bracket the same) and then **multiply by the derivative of what is inside the bracket**"

$$\frac{dy}{dx} = 5(3x - 2)^4(3)$$

Let's re-order and simplify since we can multiply in any order.

$$\frac{dy}{dx} = 5(3)(3x - 2)^4$$

$$\boxed{\frac{dy}{dx} = 15(3x - 2)^4}$$

$$y = 3(5 + x^2)^{\frac{3}{2}}. \text{ Find } \frac{dy}{dx}.$$

$$y = 3(5 + x^2)^{\frac{3}{2}}$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the harder powers type. The rule for harder powers is:

"**bring the power down to the front** (and multiply it by the number at the front if there is one), then **subtract one from the power** (keeping what is inside the bracket the same) and then **multiply by the derivative of what is inside the bracket**"

$$\frac{dy}{dx} = 3\left(\frac{3}{2}\right)(5 + x^2)^{\frac{1}{2}}(2x)$$

Let's re-order and simplify since we can multiply in any order

$$\frac{dy}{dx} = 3\left(\frac{3}{2}\right)(2)(x)(5 + x^2)^{\frac{1}{2}}$$

$$\boxed{\frac{dy}{dx} = 9x(5 + x^2)^{\frac{1}{2}}}$$

$f(x) = \frac{5}{\sqrt{2-4x}}$. Find $\frac{dy}{dx}$.

Firstly, we can bring the power up using indices rules and then it is just the harder powers type of differentiation

$$f(x) = 5(2-4x)^{-\frac{1}{2}}$$

$$y = 5(2-4x)^{-\frac{1}{2}}$$

Now we can differentiate

$$f'(x) = 5\left(-\frac{1}{2}\right)(2-4x)^{-\frac{3}{2}}(-4)$$

$$\boxed{f'(x) = 10(2-4x)^{-\frac{3}{2}}}$$

Note: A lot of students will try and use quotient rule for this. It is not necessary to use quotient rule here, but you can and it would work!

$$f(x) = \frac{5}{\sqrt{2-4x}}$$

We use quotient rule when we have division of one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) (here we have a constant and a harder power type and a constant is not considered one of the necessary types, that is why is it not necessary to use quotient rule). Also note that we can even always avoid quotient rule when we do have division of one of the 6 differentiation types, by bringing the denominator up and always just using product rule, but this can require more simplification at the end. Therefore, I would suggest using Quotient rule when it is necessary.

Exponentials: Base e

$y = e^{4x}$. Find $\frac{dy}{dx}$.

$$y = e^{4x}$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials with base *e* is:

"copy the entire exponential and then multiply by the derivative of the power"

$$\frac{dy}{dx} = e^{4x}(4)$$

$$\boxed{\frac{dy}{dx} = 4e^{4x}}$$

$y = 5e^{5x}$. Find $\frac{dy}{dx}$.

$$y = 5e^{5x}$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials with base *e* is:

"copy the entire exponential and then multiply by the derivative of the power"

$$\frac{dy}{dx} = 5e^{5x}(5)$$

$$\boxed{\frac{dy}{dx} = 25e^{5x}}$$

Exponentials: Base other than e

$$y = 2^{4x} \text{ . Find } \frac{dy}{dx}.$$

$$y = 2^{4x}$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials (with a base other than e) is:

"**copy the entire exponential** and then **multiply by the derivative of the power** and also by ln of the base"

Notice the extra purple part when we have an exponential which doesn't have a base of e.

$$\frac{dy}{dx} = 2^{4x}(4)\ln 2$$

$$\boxed{\frac{dy}{dx} = (4\ln 2)(2^{4x})}$$

$$y = 5(3^{2x}). \text{ Find } \frac{dy}{dx}.$$

$$y = 5(3^{2x})$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials (with a base other than e) is:

"**copy the entire exponential** and then **multiply by the derivative of the power** and also by ln of the base "

Notice the extra purple part when we have an exponential which doesn't have a base of e.

Don't worry about the 5 at the front, that is just hanging around at the front.

$$\frac{dy}{dx} = 5(3^{2x})(2)\ln 3$$

simplify

$$\boxed{\frac{dy}{dx} = (10\ln 3)(3^{2x})}$$

Natural Logarithms:

$$y = \ln(3x + 2). \text{ Find } \frac{dy}{dx}.$$

$$y = \ln(3x + 2)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the logs type. The rule for logs is:

"**This turns into to a fraction** which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$. **Notice how the** ln **disappears**"

$$\frac{dy}{dx} = \frac{3}{3x + 2}$$

$$\boxed{\frac{dy}{dx} = \frac{3}{3x + 2}}$$

$$y = 3\ln(x^2 + 3x + 5). \text{ Find } \frac{dy}{dx}.$$

$$y = 3\ln(x^2 + 3x + 5)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the logs type. The rule for logs is:

"This turns into to a fraction which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$ **. Notice how the ln disappears"**

Don't worry about the 3 at the front, that is just hanging around at the front.

$$\frac{dy}{dx} = 3\left(\frac{2x + 3}{x^2 + 3x + 5}\right)$$

simplify

$$\boxed{\frac{dy}{dx} = \frac{6x + 9}{x^2 + 3x + 5}}$$

Trig:

$$y = \cos 3x. \text{ Find } \frac{dy}{dx}.$$

$$y = \cos(3x)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the trig type. The rule for trig is:

"Change the trig function to what it is meant to go to (keeping the angle the same) and then multiply by the derivative of the angle"

Our 6 trig functions that we have to remember are:
$sin \Rightarrow cos$
$cos \Rightarrow -sin$
$tan \Rightarrow sec^2$
$sec \Rightarrow sec\ tan$
$cosec \Rightarrow -cosec\ cot$
$cot \Rightarrow -cosec^2$

Here we have the 2nd one: $cos \Rightarrow -sin$

$$\frac{dy}{dx} = -\sin 3x\ (3)$$

simplify

$$\boxed{\frac{dy}{dx} = -3\sin 3x}$$

$$y = \sin(4x^2). \text{ Find } \frac{dy}{dx}.$$

$$y = \sin(4x^2)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the trig type. The rule for trig is:

"Change the trig function to what it is meant to go to (keeping the angle the same) and then multiply by the derivative of the angle"

Our 6 trig functions that we have to remember are:
$sin \Rightarrow cos$
$cos \Rightarrow -sin$
$tan \Rightarrow sec^2$
$sec \Rightarrow sec\ tan$
$cosec \Rightarrow -cosec\ cot$
$cot \Rightarrow -cosec^2$

Here we have the 1st one: $sin \Rightarrow cos$

$$\frac{dy}{dx} = \cos(4x^2)\,(8x)$$

simplify

$$\boxed{\frac{dy}{dx} = 8x\cos(4x^2)}$$

Inverse Trig:

$$y = \cos^{-1}(4x). \text{ Find } \frac{dy}{dx}.$$

$$y = \cos^{-1}(4x)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the inverse trig type. The rule for inverse trig is:

$$\sin^{-1} f(x) = arcsin\, f(x) \Rightarrow \frac{derivative\ of\ angle}{\sqrt{1-(angle)^2}}$$
$$\cos^{-1} f(x) = arccos\, f(x) \Rightarrow \frac{-derivative\ of\ angle}{\sqrt{1-(angle)^2}}$$
$$\tan^{-1} f(x) = arctan\, f(x) \Rightarrow \frac{derivative\ of\ angle}{1+(angle)^2}$$

Notice how the inverse trig disappears and we have a fraction.

Here we have the second type $\cos^{-1}$

$$\frac{dy}{dx} = \frac{-4}{\sqrt{1-(4x)^2}}$$

$$\boxed{-\frac{4}{\sqrt{1-16x^2}}}$$

Harder Examples

$$y = e^{\cos x}$$

Here we have a mix of 2 types (exponential and trig)

Recall the rule for exponentials:

"copy the entire exponential and then multiply by the derivative of the power"

Recall the rule for trig:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"

Our 6 trig functions that we have to remember are:

$sin \Rightarrow cos$

$\boldsymbol{cos \Rightarrow -sin}$ **(we have this one in this example)**

$tan \Rightarrow sec^2$

$sec \Rightarrow sec\ tan$

$cosec \Rightarrow -cosec\ cot$

$cot \Rightarrow -cosec^2$

We deal with the exponential first since that is the main function, but when we differentiate the power which is part of the exponential differentiation rule, we have to use our trig differentiation rule to do this

$$\frac{dy}{dx} = e^{cosx}(-\sin x)$$

$$\frac{dy}{dx} = (-\sin x)e^{\cos x}$$

$$\frac{dy}{dx} = -\sin x\, e^{\cos x}$$

$$y = (e^{4x} + 5)^6$$

Here we have a mix of 2 types (harder power and exponential)

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

Recall the rule for exponentials:

"copy the entire exponential and then multiply by the derivative of the power"

We deal with the harder power since that is the main function, but when we differentiate inside the bracket which is part of the harder power differentiation rule, we have to use our exponential differentiation rule to do this

$$\frac{dy}{dx} = 6(e^{4x} + 5)^5(4e^{4x})$$

$$\frac{dy}{dx} = 24e^{4x}(e^{4x} + 5)^5$$

$$f(x) = \sqrt{e^{2x} + e^{-2x}}$$

Firstly we need to write this as $f(x) = (e^{2x} + e^{-2x})^{\frac{1}{2}}$

Here we have a mix of 2 types (harder power and exponential)

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

Recall the rule for exponentials:

"copy the entire exponential and then multiply by the derivative of the power"

We deal with the harder power since that is the main function, but when we differentiate inside the bracket which is part of the harder power differentiation rule, we have to use our exponential differentiation rule to do this

$$f'(x) = \frac{1}{2}(e^{2x} + e^{-2x})^{-\frac{1}{2}}(2e^{2x} + (-2)e^{-2x})$$

$$f'(x) = (e^{2x} - e^{-2x})(e^{2x} + e^{-2x})^{-\frac{1}{2}}$$

$$f'(x) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

$$y = \ln(\sin x)$$

Here we have a mix of 2 types (log and trig)

Recall the rule for logs:

"This turns into to a fraction which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$. Notice how the ln disappears"

Recall the rule for trig:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"
Our 6 trig functions that we have to remember are:
$sin \Rightarrow cos$ (we have this one in this example)
$cos \Rightarrow -sin$
$tan \Rightarrow sec^2$
$sec \Rightarrow sec\ tan$
$cosec \Rightarrow -cosec\ cot$
$cot \Rightarrow -cosec^2$

We deal with the log first since that is the main function, but when we differentiate inside the argument part which is part of the log differentiation rule, we have to use our trig differentiation rules to do this

$$\frac{dy}{dx} = \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = \cot x$$

$$y = \ln(1 - 2x)^3$$

Here we have a mix of 2 types (log and harder powers)

Recall the rule for logs:

"This turns into to a fraction which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$. Notice how the ln disappears"

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

We deal with the log first since that is the main function, but when we differentiate inside the argument part which is part of the log differentiation rule, we have to use our harder power differentiation rules to do this

$$\frac{dy}{dx} = \frac{3(1 - 2x)^2(-2)}{(1 - 2x)^3}$$

$$\frac{dy}{dx} = \frac{-6(1 - 2x)^2}{(1 - 2x)^3}$$

$$\frac{dy}{dx} = -\frac{6}{1 - 2x}$$

$$f(x) = \sin^3 4x$$

This is one that students so often get wrong.

We have to first write the trig to a power in a more familiar way

$$f(x) = \sin^3 4x = (\sin 4x)^3$$

Here we have a mix of 2 types (harder power and trig)

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

Recall the rule for trig:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"
Our 6 trig functions that we have to remember are:
$sin \Rightarrow cos$ (we have this one in this example)
$cos \Rightarrow -sin$
$tan \Rightarrow sec^2$
$sec \Rightarrow sec\ tan$
$cosec \Rightarrow -cosec\ cot$
$cot \Rightarrow -cosec^2$

We deal with the harder power since that is the main function, but when we differentiate inside the bracket which is part of the harder power differentiation rule, we have to use our trig differentiation rule to do this

$$f'(x) = 3(\sin 4x)^2 (4 \cos 4x)$$

$$f'(x) = 12 \cos 4x (\sin 4x)^2$$

$$f'(x) = 12 \cos 4x \sin^2 4x$$

Product and Quotient Rule Examples

$$y = 2x(x^2 - 1)^5. \text{ Find } \frac{dy}{dx}.$$

$$y = 2x(x^2 - 1)^5$$

We must use **product rule** here since we have one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) **multiplied together** (here we have an easy power and a harder power).

Important: People often fall into the trap of not thinking that product rule is necessary here, because they fail to realise that the easy power counts as a type so we have multiplication of 2 of the types.

Way 1: Use the formula $y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$y = 2x(x^2 - 1)^5$$

Note: We call one function u and the other v. It doesn't matter which we call u or v since the formula is

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

We are just multiplying and then adding the two multiplications. Multiplication can be done in any order and so can the addition be done in any order afterwards.

Let's call the pink function u and the blue function v

$$u = 2x, v = (x^2 - 1)^5$$

We differentiate each

$$\frac{du}{dx} = 2, \frac{dv}{dx} = 5(x^2 - 1)^4(2x)$$

Plug into the formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = 2x(5)(x^2 - 1)^4(2x) + (x^2 - 1)^5(2)$$

Simplify by re-ordering the constants from each term. Let's colour code to explain this.

$$\frac{dy}{dx} = 2x(5)(x^2 - 1)^4(2x) + (x^2 - 1)^5(2)$$

$$\frac{dy}{dx} = 20x^2(x^2 - 1)^4 + 2(x^2 - 1)^5$$

Way 2: Understand what formula is telling us

$$y = 2x(x^2 - 1)^5$$

The formula basically says in English, differentiate one function at a time

(**differentiate** 1st function)(copy 2nd function)+(copy 1st function)(**differentiate** 2nd function)

$$\frac{dy}{dx} = 2(x^2 - 1)^5 + 2x(5)(x^2 - 1)^4(2x)$$

simplify

$$\boxed{\frac{dy}{dx} = 20x^2(x^2 - 1)^4 + 2(x^2 - 1)^5}$$

$y = x^2 e^x$. Find $\frac{dy}{dx}$.

$$y = x^2 e^x$$

We must use **product rule** here since we have one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) **multiplied together** (here we have an easy power and an exponential).

Way 1: Use the formula $y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$y = x^2 e^x$$

Note: We call one function u and the other v. It doesn't matter which we call u or v since the formula is

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

We are just multiplying and then adding the two multiplications. Multiplication can be done in any order and so can the addition be done in any order afterwards.

Let's call the pink function u and the blue function v

$$u = x^2, v = e^x$$

We differentiate each

$$\frac{du}{dx} = 2x, \frac{dv}{dx} = e^x$$

Plug into the formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (x^2)(e^x) + e^x(2x)$$

Simplify by re-ordering the $2x$ in the second term

$$\frac{dy}{dx} = x^2 e^x + 2xe^x$$

Way 2: Understand what formula is telling us

$$y = x^2 e^x$$

The formula basically says in English, differentiate one function at a time

(**differentiate** 1st function)(copy 2nd function)+(copy 1st function)(**differentiate** 2nd function)

$$\frac{dy}{dx} = (2x)e^x + (x^2)(e^x)$$

Simplify

$$\boxed{\frac{dy}{dx} = 2xe^x + x^2 e^x}$$

$$y = \frac{3x + 1}{2x + 1}. \text{ Find } \frac{dy}{dx}.$$

$$y = \frac{3x+1}{2x+1}$$

We must use quotient rule here since we have **division** of one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) (here we have an easy power and an easy power type)

Way 1: Use the formula $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$y = \frac{3x + 1}{2x + 1}$$

Note: We call one function u and the other v.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

This time it MATTERS which we call u or v since we are subtracting and the denominator of the original fraction is what must be squared, not the numerator. Subtraction cannot be done in any order. For example, $4 - 2$ is not the same as $2 - 4$

So we must call u the numerator (pink function) and v the denominator (blue function)

$$u = 3x + 1, v = 2x + 1$$

We differentiate each

$$\frac{du}{dx} = 3, \frac{dv}{dx} = 2$$

Plug into the formula $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(2x + 1)(3) - (3x + 1)(2)}{(2x + 1)^2}$$

Simplify the numerator

$$\frac{dy}{dx} = \frac{(6x + 3) - (6x + 2)}{(2x + 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(2x + 1)^2}$$

Way 2: Understand what formula is telling us

$$y = \frac{3x + 1}{2x + 1}$$

The formula basically says in English:

$$\frac{(copy\ denominator)(\boldsymbol{differentiate}\ numerator) - (copy\ numerator)(\boldsymbol{differentiate}\ denominator)}{(denominator)^2}$$

$$\frac{dy}{dx} = \frac{(2x + 1)(3) - (3x + 1)(2)}{(2x + 1)^2}$$

Simplify the numerator

$$\frac{dy}{dx} = \frac{(6x + 3) - (6x + 2)}{(2x + 1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{(2x + 1)^2}}$$

$$y = \frac{x^2}{\ln x}. \text{ Find } \frac{dy}{dx}$$

$$y = \frac{x^2}{\ln x}$$

We must use quotient rule here since we have **division** of one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) (here we have an easy power and a log type)

Way 1: Use the formula $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$y = \frac{x^2}{\ln x}$$

Note: We call one function u and the other v.

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

This time it MATTERS which we call u or v since we are subtracting and the denominator of the original fraction is what must be squared, not the numerator. Subtraction cannot be done in any order. For example, $4 - 2$ is not the same as $2 - 4$

So we must call u the numerator (pink function) and v the denominator (blue function)

$$u = x^2, v = \ln x$$

We differentiate each

$$\frac{du}{dx} = 2x, \frac{dv}{dx} = \frac{1}{x}$$

Plug into the formula $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\ln x \, (2x) - x^2 \left(\frac{1}{x}\right)}{(\ln x)^2}$$

Simplify the numerator

$$\frac{dy}{dx} = \frac{2x \ln x - x}{(\ln x)^2}$$

Way 2: Understand what formula is telling us

$$y = \frac{x^2}{\ln x}$$

The formula basically says in English:

$$\frac{(copy \ denominator)(\textbf{\textit{differentiate}} \ numerator) - (copy \ numerator)(\textbf{\textit{differentiate}} \ denominator)}{(denominator)^2}$$

$$\frac{dy}{dx} = \frac{(\ln x)(2x) - x^2 \left(\frac{1}{x}\right)}{(\ln x)^2}$$

Simplify the numerator

$$\frac{dy}{dx} = \frac{2x \ln x - x}{(\ln x)^2}$$

Product and Quotient - Getting Into Certain Forms

$y = (x+1)^4(2x-2)^5$. Show that $\frac{dy}{dx} = 2(9x+1)(x+1)^3(2x-2)^4$

$$y = (x+1)^4(2x-2)^5$$

We must use product rule

$$\frac{dy}{dx} = (x+1)^4(5)(2x-2)^4(2) + (2x-2)^5(4)(x+1)^3$$

Simplify by multiplying constants and re-ordering. Let's colour code this for ease of explanation.

$$\frac{dy}{dx} = (x+1)^4(5)(2x-2)^4(2) + (2x-2)^5(4)(x+1)^3$$

$$\frac{dy}{dx} = 10(x+1)^4(2x-2)^4 + 4(2x-2)^5(x+1)^3$$

To simplify further and get into the required form we must factorise by taking out what is common to both terms. Let's colour code this again for ease of explanation.

$$\frac{dy}{dx} = 10(x+1)^4(2x-2)^4 + 4(x+1)^3(2x-2)^5$$

Take out the HCF of the numbers
Take out the HCF of the pink terms (lowest power of each)
Take out the HCF of the blue terms (lowest power of each)

$$= 2(x+1)^3(2x-2)^4[5(x+1) + 2(2x-2)]$$

Notice how we subtracted the powers in order to get the powers of the terms inside the square bracket (or asked ourselves what power we need to add to the power we have outside the bracket to end up with the power we want)

Simplify what is inside the square bracket

$$= 2(x+1)^3(2x-2)^4[5x+5+4x-4]$$

Simplify again

$$= 2(x+1)^3(2x-2)^4(9x+1)$$

$$= 2(9x+1)(x+1)^3(2x-2)^4$$

$y = \frac{x^2-4x+12}{(x-3)^2}$. Show that $\frac{dy}{dx} = -\frac{2(x+6)}{(x-3)^3}$

$$y = \frac{x^2 - 4x + 12}{(x-3)^2}$$

We must use quotient rule

$$\frac{dy}{dx} = \frac{(x-3)^2(2x-4) - (x^2-4x+12)(2)(x-3)}{(x-3)^4}$$

Re-order: Bring the constant of 2 to the front of the second term in the numerator

$$\frac{dy}{dx} = \frac{(x-3)^2(2x-4) - 2(x-3)(x^2-4x+12)}{(x-3)^4}$$

We still have not achieved the required form. There are 2 ways to proceed to do this. Since the powers are small and it is easy to simplify we don't have to factorise straight away

Way 1: Factorise the numerator straight away (best method)	Way 2: Simplify the numerator first
	This is not always the best method as it can be hard to factorise the numerator after in order to cancel
$$\frac{dy}{dx} = \frac{(x-3)^2(2x-4) - 2(x-3)(x^2-4x+12)}{(x-3)^4}$$	
	$$\frac{dy}{dx} = \frac{(2x-4)(x-3) - 2(x^2-4x+12)}{(x-3)^3}$$
Take out the HCF of the pink terms in the numerator (lowest power which is 1)	Simplify the numerator by expanding the brackets

$$\frac{dy}{dx} = \frac{(x-3)[(x-3)(2x-4) - 2(x^2 - 4x + 12)]}{(x-3)^4}$$

Simplify what is inside the square bracket

$$\frac{dy}{dx} = \frac{(x-3)[2x^2 - 10x + 12 - 2x^2 + 8x - 24]}{(x-3)^4}$$

$$\frac{dy}{dx} = \frac{(x-3)[-2x - 12]}{(x-3)^4}$$

$$\frac{dy}{dx} = \frac{-2x - 12}{(x-3)^3}$$

$$\frac{dy}{dx} = -\frac{2(x+6)}{(x-3)^3}$$

$$\frac{dy}{dx} = \frac{2x^2 - 10x + 12 - 2x^2 + 8x - 24}{(x-3)^3}$$

Collect like terms

$$\frac{dy}{dx} = \frac{-2x - 12}{(x-3)^3}$$

Factorise the numerator

$$\frac{dy}{dx} = -\frac{2(x+6)}{(x-3)^3}$$

$y = \frac{\sqrt{x^2+1}}{x-1}$. Show that $\frac{dy}{dx} = \frac{-(x+1)}{(x-1)^2\sqrt{x^2+1}}$

$$y = \frac{\sqrt{x^2 + 1}}{x - 1}$$

Firstly, let's rewrite the numerator using an exponent

$$y = \frac{(x^2 + 1)^{\frac{1}{2}}}{x - 1}$$

We must use the quotient rule

$$\frac{dy}{dx} = \frac{(x-1)\left(\frac{1}{2}\right)(x^2+1)^{-\frac{1}{2}}(2x) - (1)(x^2+1)^{\frac{1}{2}}}{(x-1)^2}$$

Simplify the numerator by multiplying constants in first term and rewriting $(x^2 + 1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} - \sqrt{x^2+1}}{(x-1)^2}$$

We now need to deal with the fraction inside the fraction.

Way 1:	**Way 2: Quickest Method**
Work on each of the numerators and denominators separately	Multiply every term in the numerator and denominator by $\sqrt{x^2 + 1}$ to kill the fractions
$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} - \sqrt{x^2+1}}{(x-1)^2}$$	Note: This is just multiplying through by $\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$
We get a common denominator in the numerator	$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} \times \sqrt{x^2+1} - \sqrt{x^2+1} \times \sqrt{x^2+1}}{(x-1)^2\sqrt{x^2+1}}$$
$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} - \frac{x^2+1}{\sqrt{x^2+1}}}{(x-1)^2}$$	$$\frac{dy}{dx} = \frac{x(x-1) - (x^2+1)}{(x-1)^2\sqrt{x^2+1}}$$
Combine fractions in the numerator	Expand the numerator
$$\frac{dy}{dx} = \frac{\frac{x(x-1) - (x^2+1)}{\sqrt{x^2+1}}}{(x-1)^2}$$	$$\frac{dy}{dx} = \frac{x^2 - x - x^2 - 1}{(x-1)^2\sqrt{x^2+1}}$$

Simplify the numerator

$$\frac{dy}{dx} = \frac{\frac{-x-1}{\sqrt{x^2+1}}}{(x-1)^2}$$

Rewrite the fraction as numerator ÷ denominator

$$\frac{dy}{dx} = \frac{-x-1}{\sqrt{x^2+1}} \div (x-1)^2$$

"keep change flip"

$$\frac{dy}{dx} = \frac{-x-1}{\sqrt{x^2+1}} \times \frac{1}{x+\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{-(x+1)}{(x-1)^2\sqrt{x^2+1}}$$

Combine like terms in the numerator

$$\frac{dy}{dx} = \frac{-(x+1)}{(x-1)^2\sqrt{x^2+1}}$$

$y = \ln(x + \sqrt{x^2+1})$. Show that $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

We must use the log differentiation rule. This turns into to a fraction which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$.

$$\frac{dy}{dx} = \frac{1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{x + \sqrt{x^2+1}}$$

Multiplying the constants in the second term in the numerator and rewrite $\sqrt{x^2+1}$ as $(x^2+1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

Way 1:	**Way 2:**
Work on each of the numerators and denominators separately	Multiply ALL terms by $\sqrt{x^2+1}$ to "kill" the fraction
$$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$	$$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$
We get a common denominator in the numerator	$$\frac{dy}{dx} = \frac{1\sqrt{x^2+1} + \frac{x}{\sqrt{x^2+1}}\sqrt{x^2+1}}{x\sqrt{x^2+1} + \sqrt{x^2+1}\sqrt{x^2+1}}$$
$$\frac{dy}{dx} = \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$	$$\frac{dy}{dx} = \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}(x + \sqrt{x^2+1})}$$
Combine fractions in the numerator	
$$\frac{dy}{dx} = \frac{\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$	"Cancel" common factors
Rewrite the fraction as numerator ÷ denominator	$$\frac{dy}{dx} = \frac{\cancel{\sqrt{x^2+1} + x}}{\sqrt{x^2+1}\cancel{(x + \sqrt{x^2+1})}}$$
$$\frac{dy}{dx} = \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} \div x + \sqrt{x^2+1}$$	$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$
"keep change flip"	
$$\frac{dy}{dx} = \frac{\cancel{x+\sqrt{x^2+1}}}{\sqrt{x^2+1}} \times \frac{1}{\cancel{x+\sqrt{x^2+1}}}$$	
"Cancel" common factors	

$$\frac{dy}{dx} = \frac{\textcolor{red}{x + \sqrt{x^2 + 1}}}{\sqrt{x^2 + 1}} \times \frac{1}{\textcolor{red}{x + \sqrt{x^2 + 1}}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

$y = \frac{x-1}{\sqrt{x+1}}$ Show that $\frac{dy}{dx} = \frac{x+c}{k\sqrt{(x+1)^p}}$, where $c, k, p \in \mathbb{N}$

$$y = \frac{x - 1}{(x + 1)^{\frac{1}{2}}}$$

Use Quotient Rule

$$\frac{dy}{dx} = \frac{(x + 1)^{\frac{1}{2}}(1) - (x - 1)\left(\textcolor{red}{\frac{1}{2}}\right)(x + 1)^{-\frac{1}{2}}}{x + 1}$$

Reorder the constants in the numerator

$$\frac{dy}{dx} = \frac{(x + 1)^{\frac{1}{2}} - \frac{1}{2}(x - 1)(x + 1)^{-\frac{1}{2}}}{x + 1}$$

To get the form that the answer wants we need to factorise the numerator by taking out what is common, which is $x + 1$ and the lowest power is $-\frac{1}{2}$

$$\frac{dy}{dx} = \frac{(x + 1)^{-\frac{1}{2}}\left[(x + 1) - \frac{1}{2}(x - 1)\right]}{x + 1}$$

Notice how we subtracted the powers in order to get the powers of the terms inside the square bracket (or asked ourselves what power we need to add to the power we have outside the bracket to end up with the power we want)

Simplify what is inside the square brackets

$$\frac{dy}{dx} = \frac{(x + 1)^{-\frac{1}{2}}\left[x + 1 - \frac{1}{2}x + \frac{1}{2}\right]}{x + 1}$$

Simplify again

$$\frac{dy}{dx} = \frac{(x + 1)^{-\frac{1}{2}}\left[\frac{1}{2}x + \frac{3}{2}\right]}{x + 1}$$

Use indices rules on the common terms. The best way to think of this is that anytime we move a term between the numerator and denominator we change the sign of the power

$$\frac{dy}{dx} = \frac{\frac{1}{2}x + \frac{3}{2}}{(x+1)(x+1)^{-\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x + \frac{3}{2}}{(x + 1)^{1+\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x + \frac{3}{2}}{(x + 1)^{\frac{3}{2}}}$$

Use indices rule $x^{\frac{a}{b}} = \sqrt[b]{x^a}$. Note when $b = 2$ we don't write it.

$$\frac{dy}{dx} = \frac{\frac{1}{2}x + \frac{3}{2}}{\sqrt{(x+1)^3}}$$

We multiply all terms by 2 to get rid of the fractions i.e. we multiply by $\frac{2}{2}$

$$\frac{dy}{dx} = \frac{x+3}{2\sqrt{(x+1)^3}}$$

$$\therefore c = 3, k = 2, p = 3$$

Parametric Differentiation

$$x = t^3 - 8t, y = t^2. \text{ Find } \frac{dy}{dx}$$

Now we have 2 equations and 3 variables. Each equation is in terms of the same extra variable. We differentiate each equation separately (using one of the 6 differentiation types which you should make sure you are good at before you proceed with this topic – see the worksheet differentiation advanced rules if not) and then plug into the following formula

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Think of the fractions on the right cross cancelling to give us what we want ultimately which is $\frac{dy}{dx}$

So, we just multiply the derivative of the y function by the "flip" of the derivative of the x function. We flip the x function because we want dx at the bottom.

Note: The formula can also be written as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Dividing fractions means to "keep change flip", so it is the same thing as the first formula

We look at each function one at a time

$x = t^3 - 8t$	$y = t^2$
Differentiate	Differentiate
$\frac{dx}{dt} = 3t^2 - 8$	$\frac{dy}{dt} = 2t$
Flip	
$\frac{dt}{dt} = \frac{1}{3t^2 - 8}$	

Formula: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

The formula just says the derivative of the y function times the "flip" of the derivative of the x function

Plugging into the formula gives

$$\frac{dy}{dx} = 2t \times \frac{1}{3t^2 - 8}$$

$$\boxed{\frac{dy}{dx} = \frac{2t}{3t^2 - 8}}$$

$x = 2t + 5, y = 3 + \frac{4}{t}$. Find $\frac{dy}{dx}$ at the point with coordinates (9,5)

Formula: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

The formula just says the derivative of the y function times the "flip" of the derivative of the x function

We look at each function one at a time

$x = 2t + 5$	$y = 3 + \dfrac{4}{t}$
Differentiate	$= 3 + 4t^{-1}$
$\dfrac{dx}{dt} = 2$	Differentiate
Flip	$\dfrac{dy}{dt} = -\dfrac{4}{t^2}$
$\dfrac{dt}{dx} = \dfrac{1}{2}$	

Plugging into the formula $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ gives

$$\frac{dy}{dx} = -\frac{4}{t^2} \times \frac{1}{2}$$

$$\frac{dy}{dx} = -\frac{2}{t^2}$$

Here we have the derivative in terms of t, HOWEVER the point given to us is in x and y coordinates. Use x AND y to solve for t and proceed as above.

We are given the coordinate $(9, 5)$ which means $x = 9$ and $y = 5$

$x = 2t + 5$	$y = 3 + \dfrac{4}{t}$
When $x = 9$	**When $y = 5$**
$2t + 5 = 9$	$3 + \dfrac{4}{t} = 5$
$t = 2$	$t = 2$

Note: often, the value(s) of t found in each equation are the same, but if you have options for t, use the value(s) of t that are the same for both x and y

Here we use $t = 2$

$$\frac{dy}{dx} = -\frac{2}{t^2}$$

When $t = 2$,

$$\frac{dy}{dx}\Big|_{t=2} = -\frac{2}{2^2}$$

$$\frac{dy}{dx}\Big|_{t=2} = -\frac{1}{2}$$

Therefore at the point (9, 5):

$$\frac{dy}{dx} = -\frac{1}{2}$$

$x = \tan t$, $y = \sin t$. Find $\frac{dy}{dx}$ at the point with coordinates $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$

Formula: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

The formula just says the derivative of the y function times the "flip" of the derivative of the x function

We look at each function one at a time

$x = \tan t$	$y = \sin t$
Differentiate	Differentiate
$\dfrac{dx}{dt} = \sec^2 t$	$\dfrac{dy}{dt} = \cos t$
$= \dfrac{1}{\cos^2 t}$	
Flip	
$\dfrac{dt}{dx} = \cos^2 t$	

Plugging into the formula $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ gives

$$\frac{dy}{dx} = \cos t \times \cos^2 t$$

$$\frac{dy}{dx} = -\cos^3 t$$

Here we have the derivative in terms of t, HOWEVER the point given to us is in x and y coordinates. Use x AND y to solve for t and proceed as above.

We are given the coordinate $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$ which means $x = \sqrt{3}$ and $y = \frac{\sqrt{3}}{2}$

$x = \tan t$	$y = \sin t$
When $x = \sqrt{3}$	**When $y = \dfrac{\sqrt{3}}{2}$**
$\tan t = \sqrt{3}$	$\sin t = \dfrac{\sqrt{3}}{2}$
$t = \dfrac{\pi}{3}, \dfrac{4\pi}{3}, \dots$	$t = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dots$

Note: often, the value(s) of t found in each equation are the same, but if you have options for t, use the value(s) of t that are the same for both x and y

Therefore here we must use the common solution $t = \frac{\pi}{3}$

$$\frac{dy}{dx} = \cos^3 t$$

When $t = \frac{\pi}{3}$,

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \cos^3\left(\frac{\pi}{3}\right)$$

$$= \left(\frac{1}{2}\right)^3$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{1}{8}$$

At the point $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$:

$$\frac{dy}{dx} = \frac{1}{8}$$